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Hypercubes and compromise values for cooperative fuzzy games

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Abstract

For cooperative fuzzy games with a non-empty core hypercubes catching the core, the Weber set and the path solution cover are introduced. Using the bounding vectors of these hypercubes, compromise values are defined. Special attention is given to the relations between these values for convex fuzzy games.

Keywords: Game theory; Compromise values of σ-type and of τ-type; Cooperative games; Core; Fuzzy coalitions; Fuzzy games; Hypercubes; Path solutions; Weber set

1. Introduction

The theory of cooperative fuzzy games started with work of Aubin (1974, 1981), where special attention is paid to the core concept. Other interesting multi-valued solutions for cooperative fuzzy games are the Weber set, the participation monotonic allocation schemes (cf. Brânzei et al., in press), the fuzzy population monotonic allocation schemes (cf. Tsurumi et al., 2001), the fuzzy version of the Milnor set of reasonable payoffs for crisp games (Milnor, 1952) and the path solution cover, which we introduce in this paper.

Much work has been done in developing one-point solution concepts of cooperative fuzzy games. Shapley values as one-point solution concept for this kind of games are studied in Aubin (1974, 1981), Butnariu (1978), Butnariu and Klement (1993) and Tsurumi et al. (2001). In Molina and Tejada (2002) and Sakawa and Nishizaki (1994) the equalizer and the lexicographical solutions are considered. We enlarge the existing literature concerning one-point solution concepts for cooperative fuzzy games with compromise values.

In the theory of cooperative crisp games these values (cf. Tijs, 1981; Tijs and Lipperts, 1982; Tijs and Otten, 1993; Bergantiños and Massó, 1996; van den Brink, 1994, 2002; van Heumen, 1984) arise as feasible compromises between upper and lower bounds of the core. Inspired by this literature, the objectives of this paper are on one hand

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to introduce upper and lower bounds for the core, the Weber set and the path solution cover of fuzzy games, and on the other hand to define compromise values based on these bounds. Special attention will be given to relations between these bounds and compromise values for the class of convex fuzzy games, introduced in Brânzei et al. (in press).

The outline of the paper is as follows. In Section 2 we recall some notions and facts from the theory of cooperative fuzzy games. Path solutions and their convex hull, the path solution cover, are introduced in Section 3. For fuzzy games with a non-empty core, hypercubes catching the core, the Weber set and the path solution cover and related compromise values are defined and studied in Sections 4 and 5, respectively.

2. Preliminaries

Given the set $N = \{1, 2, \ldots, n\}$ of players, a *fuzzy coalition* is a vector $s \in [0,1]^N$. The ith coordinate s_i of s is called the *participation level* of player i in the fuzzy coalition s. Instead of $[0,1]^N$ we will also write $\mathscr{F}^N$ for the set of fuzzy coalitions. A *fuzzy game* with player set N is a map $v : \mathscr{F}^N \to \Re$ with the property $v(0,0,\ldots,0) = 0$. The map assigns to each fuzzy coalition s a real number $v(s)$, telling what such a coalition can achieve in cooperation. The set of fuzzy games with player set N will be denoted by FG^N. The *core* of a fuzzy game v (Aubin, 1974) is defined by

$$C(v) = \left\{ x \in \Re^N \,\middle|\, \sum_{i \in N} x_i = v(e^N), \right.$$
$$\left. \sum_{i \in N} s_i x_i \geqslant v(s) \quad \text{for each } s \in \mathscr{F}^N \right\},$$

where we use the notation e^S for $S \subset N$ for the vector with $(e^S)_i = 1$ if $i \in S$, and $(e^S)_i = 0$ if $i \in N \setminus S$. The fuzzy coalition e^N is called the grand coalition because all players are present with full participation level 1. The family of fuzzy games on N with a non-empty core is denoted by FG^N_*.

A special subclass of FG^N_* is the class of convex fuzzy games introduced in Brânzei et al. (in press). Here $v \in \mathrm{FG}^N$ is called *convex* iff v satisfies the increasing average marginal return (IAMR) property, i.e. for each $s^1, s^2 \in \mathscr{F}^N$ with $s^1 \leqslant s^2$, each $i \in N$ and all $\varepsilon_1, \varepsilon_2 \in \Re_{++}$ with $s^1_i + \varepsilon_1 \leqslant s^2_i + \varepsilon_2 \leqslant 1$ it holds that

$$\varepsilon_1^{-1}(v(s^1 + \varepsilon_1 e^i) - v(s^1)) \leqslant \varepsilon_2^{-1}(v(s^2 + \varepsilon_2 e^i) - v(s^2)). \tag{1}$$

The IAMR-property (inequality (1)) expresses the fact that for a convex fuzzy game an increase in participation level of any player in a smaller coalition yields per unit of participation level less than an increase in a bigger coalition under the condition that the reached level of participation in the first case is still not bigger than the reached participation level in the second case.

For each ordering σ on N the marginal vector $m^\sigma(v)$ for $v \in \mathrm{FG}^N$ is defined as follows. For $i = \sigma(k)$ the ith coordinate $m^\sigma_i(v)$ of $m^\sigma(v)$ is equal to

$$v\left(\sum_{r=1}^{k} e^{\sigma(r)}\right) - v\left(\sum_{r=1}^{k-1} e^{\sigma(r)}\right).$$

The *Weber set* $W(v)$ for fuzzy games (cf. Brânzei et al., in press) is defined by $W(v) = \mathrm{conv}\{m^\sigma(v) \mid \sigma \text{ is an ordering on } N\}$, the convex hull of the $n!$ marginal vectors. It is proved there (Theorem 8) that

$$C(v) = W(v) \quad \text{for each convex game } v \in \mathrm{FG}^N. \tag{2}$$

3. Path solutions and the path solution cover

Let us consider paths in the hypercube $[0,1]^N$ of fuzzy coalitions, which connect $(0,0,\ldots,0)$ with $e^N = (1,1,\ldots,1)$ in a special way.

Formally, a sequence $\pi = \langle p^0, p^1, p^2, \ldots, p^m \rangle$ of $m+1$ different points in $\mathscr{F}^N$ will be called a *path* (of length m) in $[0,1]^N$ if

(i) $p^0 = (0,0,\ldots,0)$, and $p^m = (1,1,\ldots,1)$;
(ii) $p^k \leqslant p^{k+1}$ for each $k \in \{0,1,2,\ldots,m-1\}$;
(iii) for each $k \in \{0,1,2,\ldots,m-1\}$, there is one player $i \in N$ (the acting player in point p^k) such that $(p^k)_j = (p^{k+1})_j$ for all $j \in N \setminus \{i\}$, $(p^k)_i < (p^{k+1})_i$.

For a path $\pi = \langle p^0, p^1, p^2, \ldots, p^m \rangle$ let us denote by $P_i(\pi)$ the set of points p^k, where player i is acting, i.e. where $(p^k)_i < (p^{k+1})_i$. Given a game $v \in \mathrm{FG}^N$ and a path π, the payoff vector $x^\pi(v) \in \Re^N$ corresponding to v and π has the ith coordinate

$$x_i^\pi(v) = \sum_{k:p^k \in P_i(\pi)} (v(p^{k+1}) - v(p^k)).$$

Given such a path $\langle p^0, p^1, p^2, \ldots, p^m \rangle$ of length m and $v \in \mathrm{FG}^N$, one can imagine the situation, where the players in N, starting from non-cooperation ($p^0 = 0$) arrive to full cooperation ($p^m = e^N$) in m steps, where in each step one of the players increases his participation level. If the increase in value in such a step is given to the acting player, the resulting aggregate payoffs lead to the vector $x^\pi(v) = (x_i^\pi(v))_{i \in N}$. Note that $x^\pi(v)$ is an efficient vector, i.e. $\sum_{i=1}^n x_i^\pi(v) = v(e^N)$. We call $x^\pi(v)$ a *path solution*.

Let us denote by $P(N)$ the set of paths in $[0,1]^N$. Then we denote by $P(v)$ the convex hull of the set of path solutions and call it the *path solution cover*. Hence,

$$P(v) = \mathrm{conv}\{x^\pi(v) \in \Re^n \mid \pi \in P(N)\}.$$

Note that all paths $\pi \in P(N)$ have length at least n. There are $n!$ paths with length exactly n; each of these paths corresponds to a situation where one by one the players—say in the order $\sigma(1), \sigma(2), \ldots, \sigma(n)$—increase their participation from level 0 to level 1. Let us denote such a path along n edges by π^σ. Then

$$\pi^\sigma = \langle 0, e^{\sigma(1)}, e^{\sigma(1)} + e^{\sigma(2)}, \ldots, e^N \rangle.$$

Clearly, $x(\pi^\sigma) = m^\sigma(v)$. Hence,

$$W(v) = \mathrm{conv}\{x(\pi^\sigma) \mid \sigma \text{ is an ordering on } N\} \subset P(v).$$

In Brânzei et al. (in press) it was proved that the core of a fuzzy game is a subset of the Weber set. Hence

Proposition 1. *For each $v \in \mathrm{FG}^N$ we have $C(v) \subset W(v) \subset P(v)$.*

Example 2. Let $v \in \mathrm{FG}^{\{1,2\}}$ be given by $v(s_1, s_2) = s_1(s_2)^2 + s_1 + 2s_2$ for each $s = (s_1, s_2) \in \mathscr{F}^{\{1,2\}}$ and let $\pi \in P(N)$ be the path of length 3 given by $\langle (0,0), (\frac{1}{3}, 0), (\frac{1}{3}, 1), (1,1) \rangle$. Then $x_1^\pi(v) = (v(\frac{1}{3}, 0) - v(0,0)) + (v(1,1) - v(\frac{1}{3}, 1)) = 1\frac{2}{3}$, $x_2^\pi(v) = v(\frac{1}{3}, 1) - v(\frac{1}{3}, 0) = 2\frac{1}{3}$. So $(1\frac{2}{3}, 2\frac{1}{3}) \in P(v)$. The two shortest paths of length 2 given by $\pi^{(1,2)} = \langle (0,0), (1,0), (1,1) \rangle$ and $\pi^{(2,1)} = \langle (0,0), (0,1), (1,1) \rangle$ have payoff vectors $m^{(1,2)}(v) = (1,3)$, and $m^{(2,1)}(v) = (2,2)$, respectively.

4. Hypercubes as catchers of sets of payoff vectors for fuzzy games

In this section lower and upper bounds for payoff vectors in the core, the Weber set, and the path solution cover of a fuzzy game $v \in \mathrm{FG}_*^N$ are introduced. A lower (upper) bound is a payoff vector whose ith coordinate is the payoff given to player i when a "least desirable" ("most convenient") situation for him is achieved. By using pairs consisting of a lower bound and an upper bound, we obtain hypercubes which are catchers of the core, the Weber set, and the path solution cover, respectively. Compromise values are obtained in Section 5 by taking a feasible compromise between the lower and upper bounds of the three catchers.

Formally, a hypercube in $\Re^N$ is a set of vectors of the form

$$[a,b] = \{x \in \Re^N \mid a_i \leqslant x_i \leqslant b_i \text{ for each } i \in N\},$$

where $a, b \in \Re^N$, $a \leqslant b$ (and the order $\leqslant$ is the standard partial order in $\Re^N$). The vectors a and b are called *bounding vectors* of the hypercube $[a,b]$, where, more explicitly, a is called the *lower vector* and b the *upper vector* of $[a,b]$. Given a set $A \subset \Re^N$ we say that the hypercube $[a,b]$ is a *catcher* of A if $A \subset [a,b]$, and $[a,b]$ is called a *tight catcher* of A if there is no hypercube strictly included in $[a,b]$ which also catches A.

A hypercube of reasonable outcomes for a crisp game plays a role in Milnor (1952) (cf. Gerard-Varet and Zamir, 1987) and this hypercube can be seen as a tight catcher of the Weber set for crisp games. Also in Tijs (1981) and Tijs and

Lipperts (1982) hypercubes are considered which are catchers of the core of crisp games.

The objective of this section is to introduce and study catchers of the core, the Weber set and the path solution cover for games $v \in \mathrm{FG}_*^N$.

Let us first introduce a core catcher

$$HC(v) = [l(C(v)), u(C(v))]$$

for a game $v \in \mathrm{FG}_*^N$, where for each $k \in N$:

$$l_k(C(v)) = \sup\{\varepsilon^{-1}v(\varepsilon e^k) \mid \varepsilon \in (0,1]\}$$

and

$$u_k(C(v)) = \inf\{\varepsilon^{-1}(v(e^N) - v(e^N - \varepsilon e^k)) \mid \varepsilon \in (0,1]\}.$$

Proposition 3. *For each $v \in \mathrm{FG}_*^N$ and each $k \in N$: $-\infty < l_k(C(v)) \leqslant u_k(C(v)) < \infty$ and $C(v) \subset HC(v)$.*

Proof. Let $x \in C(v)$.

(i) For each $k \in N$ and $\varepsilon \in (0,1]$ we have

$$v(e^N) - v(e^N - \varepsilon e^k) \geqslant \sum_{i\in N} x_i - \left((1-\varepsilon)x_k + \sum_{i\in N\setminus\{k\}} x_i\right) = \varepsilon x_k.$$

So $x_k \leqslant \varepsilon^{-1}(v(e^N) - v(e^N - \varepsilon e^k))$, which implies $x_k \leqslant u_k(C(v)) < \infty$.

(ii) For each $\varepsilon \in (0,1]$ we have $\varepsilon x_k \geqslant v(\varepsilon e^k)$. So

$$x_k \geqslant \sup\{\varepsilon^{-1}v(\varepsilon e^k) \mid \varepsilon \in (0,1]\} = l_k(C(v)) > -\infty.$$

By using (i) and (ii) one obtains the inequalities in the proposition and the fact that $HC(v)$ is a catcher of $C(v)$. □

Now we introduce for each $v \in \mathrm{FG}_*^N$ a fuzzy variant $HW(v)$ of the hypercube of reasonable outcomes of Milnor (1952),

$$HW(v) = [l(W(v)), u(W(v))],$$

where for each $k \in N$:

$$l_k(W(v)) = \min\{v(e^{S\cup\{k\}}) - v(e^S) \mid S \subset N \setminus \{k\}\}$$

and

$$u_k(W(v)) = \max\{v(e^{S\cup\{k\}}) - v(e^S) \mid S \subset N \setminus \{k\}\}.$$

Then we have

Proposition 4. *For each $v \in \mathrm{FG}_*^N$ the hypercube $HW(v)$ is a tight catcher of $W(v)$.*

Proof. Left to the reader. □

Theorem 5. *Let $v \in \mathrm{FG}_*^N$ be a convex game. Then $HC(v) = HW(v)$ and this hypercube is a tight catcher for $C(v) = W(v)$. Further*

$$l_k(C(v)) = v(e^k),$$
$$u_k(C(v)) = v(e^N) - v(e^{N\setminus\{k\}})$$

for each $k \in N$.

Proof. For a convex game $v \in \mathrm{FG}_*^N$ the IAMR-property implies

$$\varepsilon^{-1}(v(\varepsilon e^k) - v(0)) \leqslant v(e^k) - v(0) \quad \text{for each } \varepsilon \in (0,1]$$

and

$$v(e^k) - v(0) \leqslant v(e^S + e^k) - v(e^S) \quad \text{for each } S \subset N \setminus \{k\}.$$

The first inequality corresponds to $s^1 = s^2 = 0$, $\varepsilon_1 = \varepsilon$, and $\varepsilon_2 = 1$, while the second inequality is obtained by taking $s^1 = 0$, $s^2 = e^S$, and $\varepsilon_1 = \varepsilon_2 = 1$. So, we obtain

$$\begin{aligned} l_k(C(v)) &= \sup\{\varepsilon^{-1}v(\varepsilon e^k) \mid \varepsilon \in (0,1]\} = v(e^k) \\ &= \min\{v(e^{S\cup\{k\}}) - v(e^S) \mid S \subset N \setminus \{k\}\} \\ &= l_k(W(v)). \end{aligned}$$

Similarly, from the IAMR-property it follows

$$\begin{aligned} u_k(C(v)) &= \inf\{\varepsilon^{-1}(v(e^N) - v(e^N - \varepsilon e^k)) \mid \varepsilon \in (0,1]\} \\ &= v(e^N) - v(e^{N\setminus\{k\}}) \\ &= \max\{v(e^{S\cup\{k\}}) - v(e^S) \mid S \subset N \setminus \{k\}\} \\ &= u_k(W(v)). \end{aligned}$$

This implies that $HC(v) = HW(v)$.

That this hypercube is a tight catcher of $C(v) = W(v)$ (see (2)) follows from the facts that

$$l_k(W(v)) = v(e^k) = m_k^\sigma(v),$$
$$u_k(W(v)) = v(e^N) - v(e^{N\setminus\{k\}}) = m_k^\tau(v),$$

where σ and τ are orderings on N with $\sigma(1) = k$ and $\tau(n) = k$, respectively. □

For convex fuzzy games this theorem has consequences with respect to the coincidence of some of the compromise values, which will be introduced in the next section (see Theorem 8).

Let us call a set $[a,b]$ with $a \leqslant b$ and $a \in (\Re \cup \{-\infty\})^N$ and $b \in (\Re \cup \{\infty\})^N$ a *generalized* hypercube.

Now we introduce for $v \in \mathrm{FG}_*^N$ the generalized hypercube

$$HP(v) = [l(P(v)), u(P(v))],$$

which catches the path solution cover $P(v)$ as we see in Theorem 6(i), where for $k \in N$:

$$l_k(P(v)) = \inf\{\varepsilon^{-1}(v(s+\varepsilon e^k) - v(s)) \mid s \in \mathscr{F}^N, \\ s_k < 1,\ \varepsilon \in (0, 1-s_k]\},$$

$$u_k(P(v)) = \sup\{\varepsilon^{-1}(v(s+\varepsilon e^k) - v(s)) \mid s \in \mathscr{F}^N, \\ s_k < 1,\ \varepsilon \in (0, 1-s_k]\},$$

where $l_k(P(v)) \in [-\infty, \infty)$ and $u_k(P(v)) \in (-\infty, \infty]$. Note that $u(P(v)) \geqslant u(C(v))$, $l(P(v)) \leqslant l(C(v))$.

Theorem 6

(i) *For $v \in \mathrm{FG}_*^N$, $HP(v)$ is a catcher of $P(v)$.*

(ii) *For a convex game $v \in \mathrm{FG}_*^N$,*

$$HP(v) = [Dv(0), Dv(e^N)],$$

where $D_k v(0)$ and $D_k v(e^N)$ for each $k \in N$ are the right and left partial derivative in the direction e^k in 0 and e^N, respectively.

Proof

(i) This assertion follows from the fact that for each path π and $i \in N$,

$$x_i^\pi(v) = \sum_{k:p^k \in P_i(\pi)} (v(p^k + (p_i^{k+1} - p_i^k)e^i) - v(p^k)) \\ \leqslant \sum_{k:p^k \in P_i(\pi)} (p_i^{k+1} - p_i^k) u_i(P(v)) = u_i(P(v))$$

and, similarly,

$$x_i^\pi(v) \geqslant l_i(P(v)).$$

(ii) From the IAMR-property for a convex game it follows that

$$l_k(P(v)) = \inf\{\varepsilon^{-1}(v(\varepsilon e^k) - v(0)) \mid \varepsilon \in (0,1]\} \\ = D_k v(0)$$

and

$$u_k(P(v)) = \sup\{\varepsilon^{-1}(v(e^N) - v(e^N - \varepsilon e^k)) \mid \\ \varepsilon \in (0,1]\} = D_k v(e^N). \quad \square$$

We conclude this section with a general comment on the type and amount of information that we have considered for defining the lower and upper bounds of the three catchers. Note that the lower and upper bounds of the catcher of the Weber set are obtained by using a finite number of value differences, where only coalitions corresponding to crisp coalitions play a role. The calculation of the lower and upper bounds of the catchers of the core and of the path solution cover is based on an infinite number of value differences.

5. Compromise values for fuzzy games

In Tijs (1981) bounds for the core of a crisp game (cf. Tijs and Lipperts, 1982) were used to introduce two compromise values for such games, the σ-value and the τ-value.

Let w be a crisp game with player set N. For each $S \in 2^N \setminus \{\emptyset\}$ and $i \in S$, $M_i(w) = w(N) - w(N \setminus \{i\})$ represents the utopia payoff for player i in the grand coalition N (if player i wants more, then it is advantageous for the other players in N to throw player i out). The utopia vector $M(w) = (M_1(w), \ldots, M_n(w))$ is an upper bound of the core of w.

The *σ-value* $\sigma(w)$ of a game w with

$$\sum_{i \in N} w(\{i\}) \leqslant w(N) \leqslant \sum_{i \in N} M_i(w)$$

is the feasible compromise between the stand-alone vector $a(v) = (w(\{1\}), \ldots, w(\{n\}))$ and the utopia vector $M(w)$:

$$\sigma(w) = (1-\alpha)a(w) + \alpha M(w),$$

where $\alpha \in [0,1]$ is such that $\sum_{i \in N} \sigma_i(w) = w(N)$.

Now, $R_i(S,w) := w(S) - \sum_{j \in S\setminus\{i\}} M_j(w)$ can be seen as the remainder of $i \in S$ (the amount which remains for player i if coalition S forms and all the other players in S are given their utopia payoff), and $m_i(w) := \max_{S:i \in S} R_i(S,w)$ is called the minimal right of player i in w.

The τ-*value* $\tau(w)$ of a game w with $m(w) \leqslant M(w)$ and

$$\sum_{i\in N} m_i(w) \leqslant w(N) \leqslant \sum_{i\in N} M_i(w)$$

is the feasible compromise between the minimal-rights vector $m(w) = (m_1(w), \ldots, m_n(w))$ and the utopia vector $M(w)$:

$$\tau(w) = (1-\alpha)m(w) + \alpha M(w),$$

where $\alpha \in [0,1]$ is such that $\sum_{i\in N} \tau_i(w) = w(N)$.

Note that the stand-alone vector $a(w)$ and the minimal-rights vector $m(w)$ are lower bounds for the core of w. For a survey on compromise values for crisp games we refer to Tijs and Otten (1993).

Inspired by this work we want to introduce for fuzzy games compromise values of σ-type and of τ-type for each of the solution sets $C(v)$, $W(v)$ and $P(v)$. In the first type use is made directly of the bounding vectors of $HC(v)$, $HW(v)$ and $HP(v)$, while in the τ-type compromise values the upper vector is used together with a so-called remainder vector derived from the upper vector.

To start with the first type, consider a hypercube $[a,b]$ in $\Re^N$ and a $v \in \mathrm{FG}^N_*$ such that the hypercube contains at least one efficient vector, i.e.

$$[a,b] \cap \left\{ x \in \Re^N \mid \sum_{i=1}^{n} x_i = v(e^N) \right\} \neq \emptyset.$$

Then there is a unique point $c(a,b)$ on the line through a and b which is also efficient in the sense that $\sum_{i=1}^{n} c_i(a,b) = v(e^N)$. So $c(a,b)$ is the convex combination of a and b, which is efficient. We call $c(a,b)$ the *feasible compromise* between a and b.

Now we introduce the following three σ-like compromises for $v \in \mathrm{FG}^N_*$:

$$\mathrm{val}^{\sigma}_C(v) = c(HC(v)) = c([l(C(v)), u(C(v))]),$$

$$\mathrm{val}^{\sigma}_W(v) = c(HW(v)) = c([l(W(v)), u(W(v))]),$$

and

$$\mathrm{val}^{\sigma}_P(v) = c(HP(v)) = c([l(P(v)), u(P(v))])$$

if the generalized hypercube $HP(v)$ is a hypercube.

Note that

$$\emptyset \neq C(v) \subset HC(v) \subset HP(v) \tag{3}$$

and

$$\emptyset \neq C(v) \subset W(v) \subset HW(v), \tag{4}$$

so all hypercubes contain efficient vectors and the first two compromise value vectors are always well defined. In this paper we will not deal with properties and axiomatic characterizations of the values; for such a task Tijs (1987) can be a useful guide.

For the τ-like compromise values we need to define so-called remainder vectors with the aid of a fuzzy version of the maximal remainder map $M^v : \Re^N \to \Re^N$ for a crisp game. The latter was defined in Driessen and Tijs (1985), inspired by the work of Bennett and Wooders (1979). The fuzzy version $m^v : \Re^N \to \Re^N$ of M^v for $v \in \mathrm{FG}^N_*$ we define by

$$m^v_i(z) = \sup\left\{ s_i^{-1}\left(v(s) - \sum_{j\in N\setminus\{i\}} s_j z_j \right) \mid s \in \mathscr{F}^N,\ s_i > 0 \right\}$$

for each $i \in N$ and each $z \in \Re^N$.

The following proposition shows that m^v assigns to each upper bound z of the core (i.e. $z \geqslant x$ for each $x \in C(v)$) a lower bound $m^v(z)$ of the core, called the *remainder vector* corresponding to z.

Proposition 7. *Let $v \in \mathrm{FG}^N_*$ and let $z \in \Re^N$ be an upper bound of $C(v)$. Then $m^v(z)$ is a lower bound of $C(v)$.*

Proof. Let $i \in N$ and $x \in C(v)$. For each $s \in \mathscr{F}^N$ with $s_i > 0$ we have

$$s_i^{-1}\left(v(s) - \sum_{j\in N\setminus\{i\}} s_j z_j \right) \leqslant s_i^{-1}\left(\sum_{j\in N} s_j x_j - \sum_{j\in N\setminus\{i\}} s_j z_j \right) = x_i + s_i^{-1} \sum_{j\in N\setminus\{i\}} s_j (x_j - z_j) \leqslant x_i,$$

where the first inequality follows from $x \in C(v)$ and the second inequality from the fact that z is an upper bound for $C(v)$, and then $z \geqslant x$. Hence $m^v_i(z) \leqslant x_i$ for each $i \in N$, so $m^v(z)$ is a lower bound for $C(v)$. □

Now we are able to introduce the τ-like compromise values taking into account that all upper vectors of $HC(v)$, $HW(v)$ and $HP(v)$ are upper bounds for the core of $v \in \mathrm{FG}^N_*$ as follows from (3) and (4).

So the following definitions make sense for $v \in \mathrm{FG}^N_*$:

$$\mathrm{val}^\tau_C(v) = c([m^v(u(C(v))), u(C(v))]),$$

$$\mathrm{val}^\tau_W(v) = c([m^v(u(W(v))), u(W(v))]),$$

and

$$\mathrm{val}^\tau_P(v) = c([m^v(u(P(v))), u(P(v))])$$

if the generalized hypercube $HP(v)$ is a hypercube.

The compromise value $\mathrm{val}^\tau_C(v)$ is in the spirit of the τ-value of Tijs (1981) for crisp games, and the compromise value $\mathrm{val}^\tau_W(v)$ is in the spirit of the χ-value of Bergantiños and Massó (1996), the μ-value of van Heumen (1984) and one of the values of van den Brink (1994, 2002) for crisp games, which all three coincide.

Theorem 8. *Let $v \in \mathrm{FG}^N$ be a convex game. Then*

(i) $m^v_k(u(C(v))) = m^v_k(u(W(v))) = v(e^k)$ *for each* $k \in N$;
(ii) $\mathrm{val}^\tau_C(v) = \mathrm{val}^\sigma_C(v) = \mathrm{val}^\tau_W(v) = \mathrm{val}^\sigma_W(v)$.

Proof

(i) By Theorem 5, $u_k(C(v)) = u_k(W(v)) = v(e^N) - v(e^{N\setminus\{k\}})$ for each $k \in N$. So to prove (i), we have to show that for $k \in N$

$$m^v_k(u(C(v))) = \sup\left\{ s_k^{-1}\left(v(s) - \sum_{j\in N\setminus\{k\}} s_j(v(e^N) - v(e^{N\setminus\{j\}}))\right) \mid s \in \mathscr{F}^N, s_k. > 0\right\} = v(e^k)$$

or, equivalently, that for each $s \in \mathscr{F}^N$ with $s_k > 0$,

$$s_k v(e^k) \geqslant v(s) - \sum_{j\in N\setminus\{k\}} s_j(v(e^N) - v(e^{N\setminus\{j\}})). \tag{5}$$

Let σ be an ordering on N with $\sigma(1) = k$. Then

$$v(s) = \sum_{t=1}^{n}\left(v\left(\sum_{r=1}^{t} s_{\sigma(r)}e^{\sigma(r)}\right) - v\left(\sum_{r=1}^{t-1} s_{\sigma(r)}e^{\sigma(r)}\right)\right)$$
$$= v(s_k e^k) + \sum_{t=2}^{n}\left(v\left(\sum_{r=1}^{t} s_{\sigma(r)}e^{\sigma(r)}\right) - v\left(\sum_{r=1}^{t-1} s_{\sigma(r)}e^{\sigma(r)}\right)\right).$$

Now, note that for each $t \in \{2,\dots,n\}$ the IAMR-property implies

$$s^{-1}_{\sigma(t)}\left(v\left(\sum_{r=1}^{t-1} s_{\sigma(r)}e^{\sigma(r)} + s_{\sigma(t)}e^{\sigma(t)}\right) - v\left(\sum_{r=1}^{t-1} s_{\sigma(r)}e^{\sigma(r)}\right)\right)$$
$$\leqslant v(e^{N\setminus\{\sigma(t)\}} + 1e^{\sigma(t)}) - v(e^{N\setminus\{\sigma(t)\}}).$$

So, we obtain

$$v(s) \leqslant s_k v(e^k) + \sum_{j\in N\setminus\{k\}} s_j(v(e^N) - v(e^{N\setminus\{j\}}))$$

from which (5) follows.

(ii) Since, by (i) and Theorem 5, $l_k(C(v)) = m^v_k(u(C(v))) = v(e^k)$ for each $k \in N$, it follows that $\mathrm{val}^\tau_C(v) = \mathrm{val}^\sigma_C(v) = \mathrm{val}^\tau_W(v) = \mathrm{val}^\sigma_W(v)$. □

Remark 9. Let $v \in \mathrm{FG}^N$ be a convex game. Because $u(P(v)) \geqslant u(C(v))$, it follows easily from (i) in the proof of Theorem 8 that $m_k(u(P(v))) = v(e^k)$ for each $k \in N$. But this remainder vector is in general not equal to $Dv(0)$ (see Theorem 6), so in general $\mathrm{val}^\tau_P(v)$ and $\mathrm{val}^\sigma_P(v)$ do not coincide.

Example 10. Consider the two-person convex fuzzy game with $v(s_1,s_2) = s_1(s_2)^5$ for $(s_1,s_2) \in \mathscr{F}^{\{1,2\}}$. Then, by (2) and Theorem 5, $C(v) = W(v) = \mathrm{conv}\{m^{(1,2)}(v), m^{(2,1)}(v)\} = \mathrm{conv}\{(0,1),(1,0)\}$ and $HC(v) = HW(v) = [(0,0),(1,1)]$. Hence, $\mathrm{val}^\sigma_C(v) = \mathrm{val}^\sigma_W(v) = (\frac{1}{2},\frac{1}{2})$. Further, $\mathrm{val}^\sigma_P(v) = (\frac{1}{6},\frac{5}{6})$ because, by Theorem 6, $HP(v) = [Dv(0), Dv(e^{\{1,2\}})] = [(0,0),(1,5)]$. By Theorem 8, $\mathrm{val}^\tau_C(v) = \mathrm{val}^\tau_W(v) = (\frac{1}{2},\frac{1}{2}) =$

$\mathrm{val}^{\sigma}_{C}(v) = \mathrm{val}^{\sigma}_{W}(v)$. Further, $\mathrm{val}^{\tau}_{P}(v)$ is the compromise between $m^{v}(1,5) = (0,0)$ and $(1,5)$, so in this case also $\mathrm{val}^{\tau}_{P}(v) = \mathrm{val}^{\sigma}_{P}(v) = (\frac{1}{6}, \frac{5}{6})$.

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